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NECESSARY AND SUFFICIENT CONDITIONS FOR LATENT SEPARABILITY

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Necessary and Sufficient Conditions for Latent Separability

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Abstract This paper extends the nonparametric methods developed by Samuelson (1948), Houthakker (1950), Afriat (1973), Diewert (1973) and Varian (1982, 1983) to latently separable models. It presents necessary and sufficient empirical conditions under which data on the market behaviour of a price-taking consumer, and a hypothesised allocation across latent groups are nonparametrically consistent with latent separability (Gorman (1968, 1978), Blundell and Robin (2000)). It considers homothetic latent separability and weak separability as special cases.

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What are the observable consequences of a model of consumer behaviour in which the consumer's utility function is latently separable? The notion of latent separability was introduced in Gorman (1968) and (1978) and thoroughly developed in Blundell and Robin (2000). So far the characterisation of the conditions for latent separability has been parametric. This paper seeks necessary and sufficient nonparametric conditions for latent separability.

Definition 1. *(Blundell and Robin, 2000) A direct utility function $\mathcal{U} : \mathbf{q} \in \mathbb{R}^K_+ \rightarrow \mathcal{U}(\mathbf{q}) \in \mathbb{R}$ is said to satisfy latent separability if*

$$\mathcal{U}(\mathbf{q}) = \max_{\widetilde{\mathbf{q}}^1,...,\widetilde{\mathbf{q}}^M \in \mathbb{R}^K_+} \left\{ u\left(v^1\left(\widetilde{\mathbf{q}}^1\right),...,v^M\left(\widetilde{\mathbf{q}}^M\right)\right) \mid \sum_{m=1}^{M} \widetilde{\mathbf{q}}^m = \mathbf{q} \right\}$$

where $u, v^1\left(\widetilde{\mathbf{q}}^1\right),...,v^M\left(\widetilde{\mathbf{q}}^M\right)$ are regular utility functions.

Under a latently separable functional structure the consumer's allocation problem is as follows

$$\max_{\widetilde{\mathbf{q}}^1,...,\widetilde{\mathbf{q}}^M \in \mathbb{R}^K_+} u\left(\left(v^1\left(\widetilde{\mathbf{q}}^1\right),...,v^M\left(\widetilde{\mathbf{q}}^M\right)\right)\right)$$

subject to

$$\sum_{m=1}^{M} \mathbf{p}_t' \widetilde{\mathbf{q}}^m = x_t$$

in other words they choose both the total quantity vector $\mathbf{q}_t$ and the latent allocations $\{\widetilde{\mathbf{q}}_t^m\}^{m=1,...,M}$ in order to maximise their utility subject to a budget constraint.

Suppose that there are T observations (indexed $t = 1,...,T$) on $K-$vectors of prices and corresponding demands $\{\mathbf{p}_t, \mathbf{q}_t\}$. Let $\{\widetilde{\mathbf{q}}_t^m\}_{t=1,...,T}^{m=1,...,M}$ where $\sum_{m=1}^{M} \widetilde{\mathbf{q}}_t^m = \mathbf{q}_t$ denote a hypothesised allocation of these quantity vectors across M latent groups. When can these data and hypothesised allocations be rationalised by a latently separable structure? The following definition makes clear what is meant by rationalise in this context.

Definition 2. *A latently separable utility function rationalises the data* $\{\mathbf{p}_t, \mathbf{q}_t\}_{t=1,...,T}$ *and the allocation* $\{\widetilde{\mathbf{q}}_t^m\}_{t=1,...,T}^{m=1,...,M}$ *if* $u\left(v^1\left(\widetilde{\mathbf{q}}_t^1\right), ..., v^M\left(\widetilde{\mathbf{q}}_t^M\right)\right) \geq u\left(v^1\left(\widetilde{\mathbf{q}}^1\right), ..., v^M\left(\widetilde{\mathbf{q}}^M\right)\right)$ *for all alternative allocations* $\{\widetilde{\mathbf{q}}^m\}^{m=1,...,M}$ *such that* $\mathbf{p}_t'\mathbf{q}_t \geq \sum_{m=1}^M \mathbf{p}_t'\widetilde{\mathbf{q}}^m$.

The first Theorem presents the conditions under which there exists a latently separable model of consumer preferences which rationalises the data and the hypothesised allocation.

Theorem 1. *The following statements are equivalent:*
(U) There exists a latently separable utility function, where $u(\mathbf{v})$ *and* $v^m(\widetilde{\mathbf{q}}^m)$ *are nonsatiated, monotonic, concave and continuous functions, which rationalises the data* $\{\mathbf{p}_t, \mathbf{q}_t\}_{t=1,...,T}$ *and the allocation* $\{\widetilde{\mathbf{q}}_t^m\}_{t=1,...,T}^{m=1,...,M}$.
(A) There exist numbers $\{U_t, \lambda_t\}_{t=1,...,T}$ *and* $M-$*vectors* $\{\mathbf{V}_t, \boldsymbol{\rho}_t\}_{t=1,...,T}$ *such that for all* t, s, m

$$U_s \leq U_t + \lambda_t \boldsymbol{\rho}_t' (\mathbf{V}_s - \mathbf{V}_t) \qquad (A.1)$$

$$V_s^m \leq V_t^m + \frac{1}{\rho_t^m}\mathbf{p}_t' (\widetilde{\mathbf{q}}_s^m - \widetilde{\mathbf{q}}_t^m) \qquad (A.2)$$

(G) The data $\{\mathbf{p}_t, \widetilde{\mathbf{q}}_t^m\}_{t=1,...,T}^{m=1,...,M}$ *satisfy the Generalised Axiom of Revealed Preference (GARP) and the data* $\{\mathbf{V}_t, \boldsymbol{\rho}_t\}_{t=1,...,T}$ *also satisfy GARP for some choice of* $\{\mathbf{V}_t, \boldsymbol{\rho}_t\}_{t=1,...,T}$ *which satisfy (A.2)*

Proof
$(U) \Rightarrow (A)$: First consider the implications of optimising behaviour and the first order conditions from the consumer's problem. Continuity ensures that suitable subgradients exist such that $\boldsymbol{\nabla} u(\widetilde{\mathbf{q}}_t^m) \leq \lambda_t \mathbf{p}_t$ where $\boldsymbol{\nabla} u(\widetilde{\mathbf{q}}_t^m) = \nabla u(v_t^m) \boldsymbol{\nabla} v^m(\widetilde{\mathbf{q}}_t^m)$. Define $\lambda_t \rho_t^m = \nabla u(v_t^m)$. Then $\boldsymbol{\nabla} v^m(\widetilde{\mathbf{q}}_t^m) \leq (\rho_t^m)^{-1} \mathbf{p}_t$. Now consider the concavity conditions for this structure

$$\begin{aligned} u(\mathbf{v}_s) &\leq u(\mathbf{v}_t) + \nabla u(\mathbf{v}_t)' (\mathbf{v}_s - \mathbf{v}_t) \\ v^m(\widetilde{\mathbf{q}}_s^m) &\leq v^m(\widetilde{\mathbf{q}}_t^m) + \boldsymbol{\nabla} v^m(\widetilde{\mathbf{q}}_t^m)' (\widetilde{\mathbf{q}}_s^m - \widetilde{\mathbf{q}}_t^m) \end{aligned}$$

Substituting in $\boldsymbol{\nabla} v^m(\widetilde{\mathbf{q}}_t^m) \leq (\rho_t^m)^{-1} \mathbf{p}_t$ and $\lambda_t \rho_t^m = \nabla u(v_t^m)$ preserves the inequalities and gives

$$\begin{aligned} u(\mathbf{v}_s) &\leq u(\mathbf{v}_t) + \lambda_t \boldsymbol{\rho}_t' (\mathbf{v}_s - \mathbf{v}_t) \\ v^m(\widetilde{\mathbf{q}}_s^m) &\leq v^m(\widetilde{\mathbf{q}}_t^m) + \frac{1}{\rho_t^m}\mathbf{p}_t' (\widetilde{\mathbf{q}}_s^m - \widetilde{\mathbf{q}}_t^m) \end{aligned}$$

which are conditions $(A.1)$ and $(A.2)$.
$(A) \Rightarrow (U)$: Suppose we have numbers $\{U_t, \lambda_t\}_{t=1,...,T}$ and $M-$vectors $\{\mathbf{V}_t, \boldsymbol{\rho}_t\}_{t=1,...,T}$ such that condition (A) holds. Consider some arbitrary $\{\widetilde{\mathbf{q}}^m\}^{m=1,...,M}$ such that $\mathbf{p}_t'\mathbf{q}_t \geq \sum_{m=1}^M \mathbf{p}_t'\widetilde{\mathbf{q}}^m$. We need to show that there exists a latently separable utility function, with the stated properties such that $u\left(v^1\left(\widetilde{\mathbf{q}}_t^1\right), ..., v^M\left(\widetilde{\mathbf{q}}_t^M\right)\right) \geq u\left(v^1\left(\widetilde{\mathbf{q}}^1\right), ..., v^M\left(\widetilde{\mathbf{q}}^M\right)\right)$. Using $(A.2)$ we can construct T upper bounds on $v^m(\widetilde{\mathbf{q}}^m)$ and if we take the minimum of these as the function $v^m(\widetilde{\mathbf{q}}^m)$ then we have a piecewise linear, nonsatiated, monotonic, concave and continuous utility function

$$v^m(\widetilde{\mathbf{q}}^m) = \min_s \left\{ V_s^m + \frac{1}{\rho_s^m}\mathbf{p}_s' (\widetilde{\mathbf{q}}^m - \widetilde{\mathbf{q}}_s^m) \right\}_{s=1,...,T} \leq V_t^m + \frac{1}{\rho_t^m}\mathbf{p}_t' (\widetilde{\mathbf{q}}^m - \widetilde{\mathbf{q}}_t^m)$$

Summing this inequality over m gives

$$\boldsymbol{\rho}_t'\mathbf{V}_t - \mathbf{p}_t'\mathbf{q}_t \geq \boldsymbol{\rho}_t'\mathbf{V} - \mathbf{p}_t'\widetilde{\mathbf{q}}$$

where $\mathbf{V}_t = \left[V_t^1, ..., V_t^M\right]'$, $\mathbf{V} = \left[V^1, ..., V^M\right]'$, $(V^m = v^m(\widetilde{\mathbf{q}}^m))$ and $\boldsymbol{\rho}_t = \left[\rho_t^1, ...\rho_t^M\right]$. Then since $\mathbf{p}_t'\mathbf{q}_t \geq \mathbf{p}_t'\widetilde{\mathbf{q}}$ we have $\boldsymbol{\rho}_t'\mathbf{V}_t \geq \boldsymbol{\rho}_t'\mathbf{V}$. Using $(A.3)$ we can similarly construct the following macro-utility function

$$U(\mathbf{V}) = \min_s \{U_s + \lambda_s\boldsymbol{\rho}_s'(\mathbf{V} - \mathbf{V}_s)\}_{s=1,...,T} \leq U_t + \lambda_t\boldsymbol{\rho}_t'(\mathbf{V} - \mathbf{V}_t)$$

Since $\lambda_t\boldsymbol{\rho}_t'(\mathbf{V} - \mathbf{V}_t) \leq 0$ we have $U(\mathbf{V}) \leq U_t$ as required.
$(A) \Longleftrightarrow (G)$: Follows from Afriat's Theorem.■

One special case of particular empirical interest is the one in which the latent utility functions $v(\widetilde{\mathbf{q}}^m)$ are homothetic (Gorman (1968), Blundell and Robin (2000)).

Theorem 2. *The following statements are equivalent:*
(U) There exists a homothetically latently separable utility function, where $u(\mathbf{v})$ and $v^m(\widetilde{\mathbf{q}}^m)$ are nonsatiated, monotonic, concave and continuous functions and $v^m(\widetilde{\mathbf{q}}^m)$ are homothetic, which rationalises the data $\{\mathbf{p}_t, \mathbf{q}_t\}_{t=1,...,T}$ and the allocation $\{\widetilde{\mathbf{q}}_t^m\}_{t=1,...,T}^{m=1,...,M}$.
(A) There exist numbers $\{U_t, \lambda_t\}_{t=1,...,T}$ and $M-$vectors $\{\mathbf{V}_t, \boldsymbol{\rho}_t\}_{t=1,...,T}$ such that for all t, s, m

$$U_s \leq U_t + \lambda_t\boldsymbol{\rho}_t'(\mathbf{V}_s - \mathbf{V}_t) \quad (A.1)$$

$$V_s^m \leq V_t^m + \frac{1}{\rho_t^m}\mathbf{p}_t'(\widetilde{\mathbf{q}}_s^m - \widetilde{\mathbf{q}}_t^m) \quad (A.2)$$

$$\rho_t^m V_t^m = \mathbf{p}_t'\widetilde{\mathbf{q}}_t^m \quad (A.3)$$

(G) The data $\{\mathbf{p}_t, \widetilde{\mathbf{q}}_t^m\}_{t=1,...,T}^{m=1,...,M}$ satisfy the Homothetic Axiom of Revealed Preference (HARP) and the data $\{\mathbf{V}_t, \boldsymbol{\rho}_t\}_{t=1,...,T}$ also satisfy GARP for some choice of $\{\mathbf{V}_t, \boldsymbol{\rho}_t\}_{t=1,...,T}$ which satisfy $(A.2)$ and $(A.3)$

Proof
Analogous to Theorem 1 noting that $(A.2)$ and $(A.3)$ are necessary and sufficient for the existence of M homothetic utility functions $V^m(\widetilde{\mathbf{q}}_t^m)$ such that for any $\widetilde{\mathbf{q}}^m$ with that $\mathbf{p}_t'\widetilde{\mathbf{q}}_t^m \geq \mathbf{p}_t'\widetilde{\mathbf{q}}^m$ then $V^m(\widetilde{\mathbf{q}}_t^m) \geq V^m(\widetilde{\mathbf{q}}^m)$ (Varian (1983), Theorem 2).■

Another special case concerns the situation when the allocations of goods across latent utilities is exclusive as this corresponds to weak separability.

Theorem 3. *When the hypothesised allocations are such that $\widetilde{q}_t^{k,m} = q_t^k$ for all k and t and some m (that is the allocations to latent groups are exclusive and constant over observations), then the conditions in Theorems 1 and 2 are equivalent to those for weak separability and homothetic weak separability respectively.*

Proof
Obvious from a comparison with Varian (1983) Theorems 3 and 5. ■

I cannot think of any way in which either the number or the composition of the latent groups can be nonparametrically identified from the observed data. Instead the Theorems presented are simply a way of testing the data and any hypothesised allocation for consistency with the model. In this respect the results here are similar to Varian's (1983) for weak separability, where the number and makeup of separable subgroups are not identified from data, but where the conditions under which any hypothesised grouping are rationalisable with weak separability are set out. Note that the results here can also be applied to the analysis of production functions with a suitable change in the interpretation of the notation and by requiring the top-level data to satisfy the weak axiom of profit maximisation rather than GARP (see Varian (1984), Theorem 9, for the conditions for weak separability in production functions).

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